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On the Remainder of Laplace's Series.

By EMORY MCCLINTOCK, *Milwaukee, Wis.*

The remainder of Lagrange's series has been given by Popoff and Zolotareff* in the form $\frac{1}{n!} D^n \int_z^y (x\phi t + z - t)^n f' t \cdot dt$, where $y = z + x\phi y$, and $D = \frac{d}{dz}$.

The corresponding expression for the remainder of Laplace's series, where $y = f(z + x\phi y)$, is $r_n = \frac{1}{n!} D^n \int_{fz}^y (x\phi t + z - f^{-1} t)^n f' t \cdot dt$.

To prove this, it is only necessary to perform one of the n differentiations indicated, by the usual rule for the differentiation of definite integrals with variable limits. In this way we find that

$$r_n = r_{n-1} - \frac{1}{n!} x^n D^{n-1} (\phi f z)^n D f z,$$

whence, if $m = n - 1$

$$r_m = \frac{1}{m+1!} x^{m+1} D^m (\phi f z)^{m+1} D f z + r_{m+1}.$$

Since $r_0 = fy - f f z$, we have at once, by successive substitution,

$$fy = f f z + x \phi f z D f z + \frac{1}{2!} x^2 D (\phi f z)^2 D f z + \dots + \frac{1}{n!} x^n D^{n-1} (\phi f z)^n D f z + r_n.$$

Comparing this result with Laplace's theorem, we see that we have in r_n an expression for the remainder after $n + 1$ terms.

If $fz = z$, or, in other words, if $y = z + x\phi y$, we have M. Popoff's expression for the remainder in Lagrange's theorem. The above method of proof resembles in principle that by which Zolotareff proves the latter expression.

It must not be thought that the reasoning here employed constitutes a proof of Laplace's theorem, for evidence is still needed that $r_\infty = 0$ when the series is convergent. In other words, the series is not thus proved to be necessarily the true development of fy . Suppose $s_n = \frac{x^{n+1} - 1}{1 - x}$, so that

$$s_0 = -1 = x + x^2 + x^3 + \dots + x^n + s_n;$$

this would not furnish the true development of -1 in terms of x .

* See Williamson's *Integral Calculus*, 3d ed.; *Jahrbuch über die Fortschritte der Math.* VIII; these referring to Popoff (*Comptes Rendus*, 1861, pp. 795-8,) and Zolotareff (*Nouv. Annales*, 2d series, XV, 422-3).

The consent of the family of the late Professor Benjamin Peirce has been kindly given to the publication in the *American Journal of Mathematics*, of his valuable and unique memoir on Linear Associative Algebra, of which only a small number of copies in lithograph were taken in the author's lifetime, for distribution among his friends. This publication will, it is believed, supply a want which has been long and widely felt, and bring within the reach of the general mathematical public a work which may almost be entitled to take rank as the *Principia* of the philosophical study of the laws of algebraical operation.

Linear Associative Algebra.

A Memoir read before the National Academy of Sciences in Washington, 1870.

BY BENJAMIN PEIRCE.

WITH NOTES AND ADDENDA, BY C. S. PEIRCE, SON OF THE AUTHOR.

1. Mathematics is the science which draws necessary conclusions.

This definition of mathematics is wider than that which is ordinarily given, and by which its range is limited to quantitative research. The ordinary definition, like those of other sciences, is objective; whereas this is subjective. Recent investigations, of which quaternions is the most noteworthy instance, make it manifest that the old definition is too restricted. The sphere of mathematics is here extended, in accordance with the derivation of its name, to all demonstrative research, so as to include all knowledge strictly capable of dogmatic teaching. Mathematics is not the discoverer of laws, for it is not induction; neither is it the framer of theories, for it is not hypothesis; but it is the judge over both, and it is the arbiter to which each must refer its claims; and neither law can rule nor theory explain without the sanction of mathematics. It deduces from a law all its consequences, and develops them into the suitable form for comparison with observation, and thereby measures the strength of the argument from observation in favor of a proposed law or of a proposed form of application of a law.

Mathematics, under this definition, belongs to every enquiry, moral as well as physical. Even the rules of logic, by which it is rigidly bound, could not be deduced without its aid. The laws of argument admit of simple statement, but they must be curiously transposed before they can be applied to the living speech and verified by observation. In its pure and simple form the syllogism cannot be directly compared with all experience, or it would not have required an